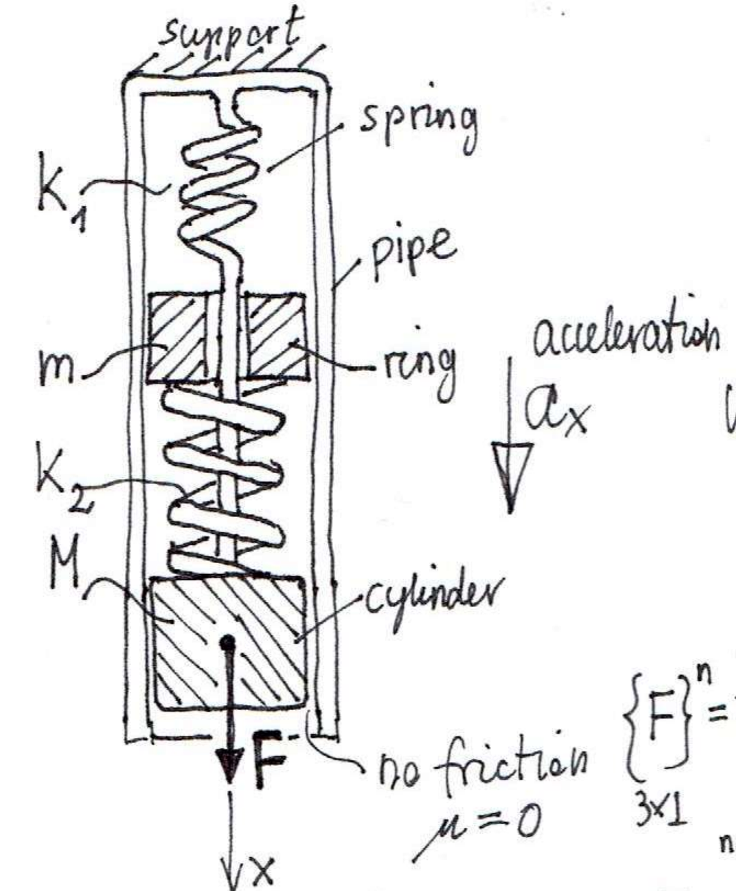
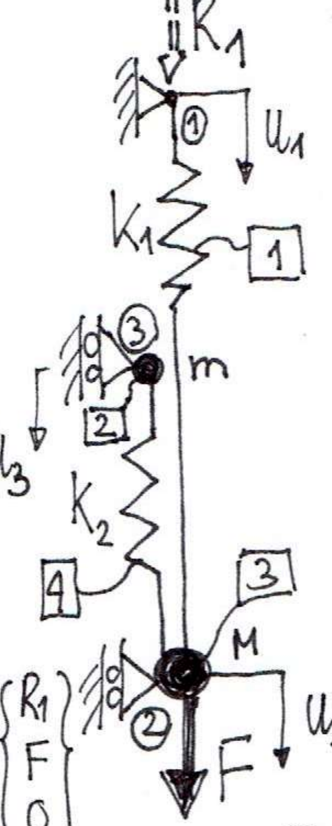
Example. Build a FE model. Find the elastic strain energy and the potential energy of loading. Calculate displacements and reactions.



(weightless springs)



□ - finite elements

○ - nodes

NOE = 4

NON = 3

$n_p = 1$

NDOF = $3 \cdot 1 = 3$

$$\underset{3\times1}{\{F\}^n} = \begin{Bmatrix} R_1 \\ F \\ 0 \end{Bmatrix}$$

no force at node 3

$$\underset{\substack{NDOF\times1 \\ \parallel \\ 3}}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$U = \frac{1}{2} \underset{1\times3}{\lfloor q \rfloor} \cdot \underset{3\times3}{[K]} \underset{3\times1}{\{q\}}, \quad W = \underset{1\times3}{\lfloor q \rfloor} \cdot \underset{3\times1}{\{F\}}$$

finite element $\boxed{1}$: ■ – local notation

①=① 

②=②

$$\underset{1\times 2}{\lfloor q \rfloor_1} = \lfloor q_1, q_2 \rfloor_1 = \lfloor u_1, u_2 \rfloor_1 \quad , \quad \underset{2\times 2}{[k]_1} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

$$\underset{1\times 2}{} \overset{\parallel}{(ne)}$$

$$\underset{1\times 3}{\lfloor q \rfloor} = \lfloor \underbrace{\boxed{u_1, u_2}}_{\lfloor q \rfloor_1}, u_3 \rfloor \qquad [k]_1^* = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lfloor F^x \rfloor_1 = \lfloor F_{11}, F_{21} \rfloor_1 = \lfloor 0, 0 \rfloor_1$$

$$\underset{1\times 3}{\lfloor F^x \rfloor_1^*} = \lfloor F_{11}, F_{21}, 0 \rfloor = \lfloor 0, 0, 0 \rfloor$$

finite element $\boxed{2}$

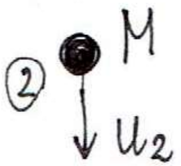
③ 

$$\underset{1\times 3}{\lfloor q \rfloor} = \lfloor \underbrace{u_1, u_2}, \boxed{u_3} \rfloor \quad ; \quad [k]_2^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F^x_2 = m \cdot a_x$$

$$\underset{1\times 3}{\lfloor F^x \rfloor_2^*} = \lfloor \overbrace{0, 0}, F_2^x \rfloor = \lfloor 0, 0, ma_x \rfloor$$

finite element $\boxed{3}$

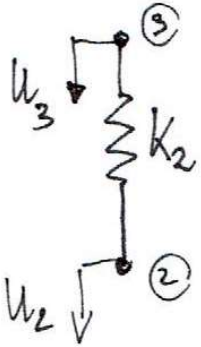


$$\underset{1\times3}{\lfloor q \rfloor} = \lfloor u_1, \boxed{u_2}, u_3 \rfloor \quad ; \quad [k]^*_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F^x_3 = M\cdot a_x$$

$$\underset{1\times3}{\lfloor F^x \rfloor^*_3} = \lfloor 0, F^x_3, 0 \rfloor = \lfloor 0, Ma_x, 0 \rfloor$$

finite element $\boxed{4}$



$$\underset{1\times2}{\lfloor q \rfloor_4} = \lfloor q_1, q_2 \rfloor_4 = \lfloor u_3, u_2 \rfloor_4 \quad ; \quad \underset{2\times2}{[k]_4} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\underset{1\times3}{\lfloor q \rfloor} = \lfloor u_1, \boxed{u_2, u_3} \rfloor$$

$$[k]^*_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

$$\underset{1\times2}{\lfloor F^x \rfloor_4} = \lfloor F_{14}, F_{24} \rfloor_4 = \lfloor 0, 0 \rfloor_4$$

$$\lfloor F^x \rfloor^*_4 = \lfloor 0, F_{24}, F_{14} \rfloor = \lfloor 0, 0, 0 \rfloor$$

global stiffness matrix:

$$\underset{3\times3}{[K]} = \sum_{e=1}^{4} [k]_e^* = \left[\begin{array}{c|c|c} k_1+0+0+0 & -k_1+0+0+0 & 0+0+0+0 \\ \hline -k_1+0+0+0 & k_1+0+0+k_2 & 0+0+0-k_2 \\ \hline 0+0+0+0 & 0+0+0-k_2 & 0+0+0+k_2 \end{array}\right] =$$

$$= \left[\begin{array}{c|c|c} k_1 & -k_1 & 0 \\ \hline -k_1 & k_1+k_2 & -k_2 \\ \hline 0 & -k_2 & k_2 \end{array}\right]$$

elastic strain energy:

$$U = \frac{1}{2} \lfloor u_1, u_2, u_3 \rfloor \cdot \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

(we need u_2 and u_3 $(u_1=0)$ to find the value of U.)

$$\lfloor F \rfloor_e = \lfloor F^x \rfloor_e + \cancel{\lfloor F^p \rfloor_e} = \lfloor F^x \rfloor_e \Rightarrow \lfloor F \rfloor_e^* = \lfloor F^x \rfloor_e^*$$

(no surface load)

$$W = \underset{1\times 3}{\lfloor q \rfloor} \cdot \underset{3\times 1}{\{F\}}$$

$$W = \underset{1\times 3}{\lfloor q \rfloor} \cdot \left(\sum_{e=1}^{4} \{F\}_e^* + \{F\}^n \right) = \underset{1\times 3}{\lfloor q \rfloor} \cdot \left(\begin{Bmatrix} 0+0+0+0 \\ 0+0+Ma_x+0 \\ 0+ma_x+0+0 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ F \\ 0 \end{Bmatrix} \right) =$$

$$= \lfloor u_1, u_2, u_3 \rfloor \cdot \begin{Bmatrix} R_1 \\ Ma_x + F \\ ma_x \end{Bmatrix} \underset{(u_1=0)}{=\!=} u_2 \cdot (Ma_x + F) + u_3 \cdot ma_x$$

(we need u_2 and u_3 to find the value of W.)

Solution

$$\boxed{V = \frac{1}{2}\lfloor q \rfloor [K] \cdot \{q\} - \lfloor q \rfloor \cdot \{F\}} \longrightarrow \min \longrightarrow \underset{3\times3}{[K]} \cdot \underset{3\times1}{\{q\}} = \underset{3\times1}{\{F\}}$$

$$\boxed{u_1 = 0} \longrightarrow \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ Ma_x + F \\ ma_x \end{Bmatrix}$$

$$\underset{2\times2}{[K]} \cdot \underset{2\times1}{\{q\}} = \underset{2\times1}{\{F\}} \longleftarrow$$

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} Ma_x + F \\ ma_x \end{Bmatrix}$$

$$\begin{cases} (k_1+k_2)\cdot u_2 - k_2 \cdot u_3 = Ma_x + F \\ -k_2 \cdot u_2 + k_2 \cdot u_3 = ma_x \end{cases} \Longrightarrow u_2,\ u_3$$

reaction:

$$\left(\text{1st row of } \underset{3\times3}{[K]}\right) \cdot \underset{3\times1}{\{q\}} = R_1 \implies R_1 = k_1 \cdot 0 - k_1 \cdot u_2 + 0 \cdot u_3 = -k_1 u_2$$